Pergamon

Appl. Math. Lett. Vol. 7, No. 6, pp. 81–84, 1994

0893-9659/94 \$7.00 + 0.00

0893-9659(94)00096-4

The Derivative of a Continuous Nonconstant Function

M. MARTELLI AND M. DANG
Mathematics Department, CSU Fullerton
Fullerton, CA 92634, U.S.A.

(Received February 1994; accepted March 1994)

Abstract—Using elementary ideas and techniques, we prove (Theorem 2) that for a nonconstant differentiable function $f : [a,b] \to \mathbb{R}$, $a < b$, the set $S(f) = \{x \in [a,b] : f'(x) \neq 0\}$ cannot be negligible. This result remains valid if f fails to be differentiable on a countable subset of [a,b].

Keywords—Derivative, Countable, Uncountable, Covering, Negligible.

Calculus students are familiar with the following result: *A function f, continuous on $[a,b]$ and differentiable on (a,b), is constant on the interval $[a,b]$ if and only if $f'(x) = 0$ for every $x \in (a,b)$.* Thus, for a function f, which is differentiable on (a,b) and nonconstant on $[a,b]$, the set $S(f) = \{x \in (a,b) : f'(x) \neq 0\}$ is not empty.

Being "not empty" is certainly an important property of $S(f)$. By asking appropriate questions, we can find additional interesting information about $S(f)$. One such question could be: *is there any function f, differentiable on (a,b) and nonconstant on $[a,b]$ such that $S(f) = \{x \in (a,b) : f'(x) \neq 0\}$ is finite or countable*? The answer to this question can be derived from the following result.

THEOREM 1. *Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then the image of f', $\operatorname{Im} f'$, is an interval.*

PROOF. It is enough to show that if $f'(x)$ changes sign in (a,b), then it must be equal to zero at some point of (a,b). Assume that $f'(x_1)f'(x_2) < 0$ where $x_1, x_2 \in (a,b)$ and $x_1 < x_2$. Then, in the case when $f'(x_1) < 0 < f'(x_2)$, we obtain that x_1 and x_2 are both maximum points of f in $I = [x_1, x_2]$. Hence, f has an absolute minimum point $x_0 \in (x_1, x_2)$, and $f'(x_0) = 0$. The proof for the case $f'(x_2) < 0 < f'(x_1)$ is similar.

We now use Theorem 1 to prove that the answer to our question is negative. In fact, assume that $S(f)$ is either finite or countable. Then $f'(S)$ cannot be uncountable. Since $\operatorname{Im} f' \subset \{0\} \cup f'(S)$ and, by Theorem 1, $\operatorname{Im} f'$ is an interval, we obtain that $\operatorname{Im} f'$ must be a point. This implies the existence of $d \neq 0$ such that $f'(x) = d$ for all $x \in [a,b]$. Hence, $S(f) = [a,b]$. This contradicts our previous assumption.

Another interesting question about $S(f)$ could be the following: *given an uncountable set $K \subset [a,b]$, can we find a function f, which is differentiable on (a,b) and nonconstant on $[a,b]$ such that $S(f) = K$?* Also the answer to this question is negative. In fact, Theorem 2 below says

The authors thank C. Coleman of Harvey Mudd College, and S. Grabiner of Pomona College for several helpful conversations on the subject matter of this paper.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX that $S(f)$ is not negligible. This result can be obtained in many different ways, and the main goal of this paper is to provide a proof by means which are easily accessible to senior mathematics majors.

We first present some definitions and notations which will be used in the proof. Recall that a set $N \subset [a,b]$ is **negligible** (or **null**) if, for every $r > 0$, we can find a cover of N with a countable family of open intervals of total length not exceeding r. Every finite or countable set is clearly negligible. Thus, nonnegligible sets must be uncountable. However, not every uncountable set is nonnegligible. For example, the Cantor ternary set is uncountable but negligible. All intervals with more than one point are uncountable and nonnegligible.

The following lemma contains some elementary properties of negligible sets. The length of a bounded interval I is denoted by $L(I)$. The complement of a set $A \subset \mathbb{R}$ is denoted by A^c, i.e., $A^c = \{x \in \mathbb{R} : x \notin A\}$.

LEMMA 1.

(a) *Any subset of a negligible set is itself negligible.*
(b) *Any countable set is negligible.*
(c) *Let N be a negligible set and I be an interval, $L(I) > 0$. Then $I \cap N^c \neq \emptyset$.*
(d) *Let $N_1, \dots N_i, \dots$ be a countable family of negligible sets. Then $N = \cup_i N_i$ is negligible.*

PROOF. The first three properties can easily be established from the definition of a negligible set. To prove the last property, let $r > 0$ be given. For each N_i, find a countable family $F_i = \{J_{i_1}, \dots, J_{i_n}, \dots\}$ of open intervals such that

$$N_i \subset \bigcup_k J_{i_k} \quad \text{and} \quad \sum_{k=1}^{\infty} L(J_{i_k}) \leq \frac{r}{2^i}.$$

The family $F = \{J_{i_k} : k = 1, 2, \dots, n, \dots; i = 1, 2, \dots, m, \dots\}$ is a countable covering of N, and the total length of all intervals of F does not exceed

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} L(J_{i_k}) = \sum_{i=1}^{\infty} \frac{r}{2^i} = r.$$

Therefore, $N = \cup_i N_i$ is negligible.

We now state and prove the result we announced before.

THEOREM 2. *Let a nonconstant function $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $[a,b]$ except possibly on a countable set C. Then $S(f) = \{x \in [a,b] \setminus C : f'(x) \neq 0\}$ is not negligible.*

To prove Theorem 2, we need two auxiliary lemmas.

LEMMA 2. *Assume that $f : [a,b] \to \mathbb{R}$ is differentiable on a negligible set $Y \subset [a,b]$ except possibly on a countable set $C \subset Y$. Then $f(Y)$ is negligible.*

PROOF. Since $f(C)$ is countable, and therefore, negligible (see Lemma 1), we may assume, without loss of generality, that f is differentiable at every point of Y. Define $Y_k, k = 1, 2, \dots$ to be the negligible subsets of $Y : Y_k = \{x \in Y : |f'(x)| \leq k\}$. Then $Y = \cup_k Y_k$ and it is enough to show that $f(Y_k)$ is negligible for every k.

Let k be fixed and set $X = Y_k$. Select $x \in X$. The inequality $|f'(x)| \leq k$ implies the existence of $d(x) > 0$ such that $|f(y) - f(x)| \leq (k+1)|y - x|$ for all $x \in [a,b]$, $|y - x| \leq d(x)$. For every $m = 1, 2, \dots$ define

$$X_m = \left\{x \in X : |f(y) - f(x)| \leq (k+1)|y-x|, \text{ for all } y \in \left(x - \frac{1}{m},\ x + \frac{1}{m}\right) \cap [a,b]\right\}.$$

Since $X = \cup_m X_m$ it is enough to prove that $f(X_m)$ is negligible for every m.

Let m be fixed and $r < 1/m$ be given. Cover X_m with a countable family of open intervals $\{I_{m_i} : i = 1, 2, \dots\}$ of total length not exceeding $r/3(k+1)$. Each interval I_{m_i} has the property $L(I_{m_i}) < 1/m$. Let $x_i \in I_{m_i}$. From the definition of X_m, we derive $|f(y) - f(x_i)| \leq (k+1)|y - x_i|$, for every $y \in I_{m_i}$. Consequently, $L(f(I_{m_i})) \leq 2(k+1)L(I_{m_i})$ and $f(X_m)$ can be covered by a countable family of open intervals of total length smaller than

$$3(k+1)\sum_{i=1}^{\infty} L(I_{m_i}) \leq 3(k+1)\frac{r}{3(k+1)} = r.$$

Since this is true for every r, we obtain that $f(X_m)$ is negligible.

The assumption of differentiability is crucial in Lemma 2 since we can find a continuous function f on $[0,1]$ which maps the Cantor set onto the interval $[0,1]$.

The next lemma can be seen as a generalization of the fact that if f' is identically zero then $\operatorname{Im} f$ is just one point.

LEMMA 3. *Let $f : [a,b] \to \mathbb{R}$ and $Z = \{x \in [a,b] : f'(x) = 0\}$. Then $f(Z)$ is negligible.*

PROOF. Without loss of generality, we may assume that $a = 0$ and $b = 1$. Let $r > 0$ be given and for every positive integer n define

$$Z_n = \left\{ z \in Z : |f(z) - f(x)| \leq \frac{r}{2}\ |x - z|, \quad \text{for every } z \in \left(x - \frac{1}{2^n},\ z + \frac{1}{2^n}\right) \cap [0,1] \right\}.$$

Obviously, $Z = \cup_{n=1}^{\infty} Z_n$. Let $n_1 \geq 1$ be the first index such that Z_{n_1} is not empty. Partition $[0,1] = J_1$ into 2^{n_1} intervals with the equally spaced nodes $0, 1/2^n, \dots, 1$. Denote the 2^{n_1} consecutive intervals by $I_m, m = 1, 2, \dots, 2^{n_1}$ and let k_1 be the set of indices for which $I_m \cap Z_{n_1} \neq \emptyset$.

For every $m \in k_1$ select $z_m \in I_m \cap Z_{n_1}$. From the definition of Z_{n_1}, we derive

$$|f(x) - f(z_m)| \leq \frac{r}{2}\,|x - z_m|, \qquad \text{for every } x \in \left(x - \frac{1}{2^n}, z + \frac{1}{2^n}\right) \cap [0,1].$$

Hence, $f(Z_{n_1})$ can be covered by k_1 open intervals of total length not exceeding $rk_1 2^{-n_1}$.

Let J_2 be the complement in $[0,1]$ of $\cup\{I_m : m \in k_1\}$ and let $n_2 > n_1$ be the first index such that $Z_{n_2} \cap J_2 \neq \emptyset$. Cover J_2 with nonoverlapping intervals I_m of length 2^{-n_2}. Let k_2 be the set of indices for which $I_m \cap Z_{n_2} \neq \emptyset, m \in k_2$. With the same argument used above for k_1, we establish that $f(Z_{n_2})$ can be covered by k_2 open intervals of total length not exceeding $rk_2 2^{-n_2}$. Moreover, $k_1 2^{-n_1} + k_2 2^{-n_2} \leq 1$. This process can be continued to find $k_3, \dots, k_p, \dots$ such that

$$k_1 2^{-n_1} + k_2 2^{-n_2} + \cdots + k_p 2^{-n_p} + \cdots \leq 1.$$

Hence, $f(Z)$ can be covered with a countable family of open intervals of total length not exceeding r. Since this is true for every r, we obtain that $f(Z)$ is negligible.

PROOF OF THEOREM 2. Since f is continuous, $\operatorname{Im} f$ is a closed interval J. Assume that $S = S(f)$ is negligible and let $Z = \{x \in [a,b] : f'(x) = 0\}$.

We have $[a,b] = C \cup Z \cup S$ and we know that $f(C), f(S)$ and $f(Z)$ are all negligible. From Lemma 1 we obtain that $\operatorname{Im} f$ must be a point. Hence f is constant. This contradiction establishes the theorem.

Other proofs of Theorem 2 can be provided using known theorems of the Lebesgue theory of Measure and Integration. However such proofs require a breadth of knowledge not usually expected from an undergraduate and are based on a platform of results more sophisticated and complex than our Lemmas (see [2]).

As we mentioned before, there exists a continuous function f which maps the Cantor set K onto the interval $[0,1]$. Moreover f can be constructed so that $f'(x) = 0$ for every $x \notin K$. From Theorem 2, we derive that f' cannot exists on an uncountable set $K' \subset K$ (see [4]).

A question of some interest, and related to the topics of this paper, is the construction of a negligible and uncountable set K which is dense in $[0,1]$. This can be accomplished as follows. Let $T = \{q_1, q_2, \dots\}$ be a listing of all rationals of $[0,1]$ and let K be the Cantor set. It is known that K is negligible and uncountable (see [1]) . Define

$$K_n = q_n + K(\operatorname{mod} 1) = \begin{cases} q_n + c, & \text{if } q_n + c < 1, \\ q_n + c - 1, & \text{if } q_n + c \geq 1, \end{cases} \quad \text{for every } c \in K.$$

Then $K_\infty = \cup_n K_n$ is dense in $[0,1]$, negligible, and uncountable.

REFERENCES

1. R.P. Boas, Jr., *A Primer of Real Functions*, The Carus Mathematical Monographs #13 MAA, J. Wiley and Sons, (1961).
2. A.M. Bruckner, Differentiation of real functions, *Lecture Notes in Mathematics, No. 659*, Springer-Verlag, (1978).
3. A.C.M. van Rooij and W.H. Schikhof, *A Second Course on Real Functions*, Cambridge University Press, (1982).
4. R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.* **53**, 427–439 (1943).